Ph.D. QUALIFYING EXAMINATION – PART A

Tuesday, January 15, 2019, 1:00 – 5:00 P.M.

Work each problem on a separate sheet(s) of paper and put your identifying number on each page. Do not use your name. Each problem has equal weight. A table of integrals can be used. Some physical constants and mathematical definitions will be provided if needed.

A1. A solid cylinder of mass M and radius R, rotating with angular speed ω_0 about an axis through its center, is set on a horizontal surface for which the kinetic friction coefficient is μ_k. Acceleration due to gravity is g.

(a) Calculate the linear acceleration of the center of mass, and the angular acceleration of rotation about the center of mass.

(b) The cylinder is initially slipping completely. Eventually, the cylinder is rolling without slipping. Calculate the distance the cylinder moves before slipping stops.

(c) Calculate the work done by the friction force on the cylinder as it moves from where it was set down to where it begins to roll without slipping.

Note: the cylinder has rotational inertia $I_{cm} = \frac{1}{2}MR^2$.

A2. A sphere of homogenous linear dielectric material is placed in an otherwise uniform electric field, which at large distances from the sphere is directed along the z axis and has magnitude E_0. The dielectric sphere has a radius R and a dielectric constant $\kappa = \varepsilon/\varepsilon_0$, where ε is the permittivity of the sphere and ε_0 is the permittivity of free space.

a) Determine the electric potential both inside and outside the dielectric sphere.

b) Determine the electric field inside the dielectric sphere in terms of κ and E_0.

c) Determine the polarization $\vec{P}$ of the dielectric sphere in terms of ε_0, κ and E_0.

d) Determine the polarization-surface-charge density σ.

Recall the general solution to Laplace's equation in spherical coordinates if there is no ϕ dependence is given by

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

A3. (a) Starting with Newton's second law, compute the work required to accelerate a particle of mass m from rest to a relativistic (comparable to speed of light c) speed v. Remember to use the relativistic expression for momentum in Newton's second law.

(b) The work found in part (a) is the relativistic kinetic energy K. Show that the total energy of the particle ($E = mc^2 + K$) and the momentum, satisfy the following equation:

$E = \sqrt{m^2c^4 + p^2c^2}$.

A4. A particle of charge e and mass m moving in one-dimension is subjected to a spatially uniform electric field of magnitude E_0 directed along the positive x-axis (i.e., the axis along which the particle moves). Let $|\varepsilon\rangle$ denote an energy eigenstate of the particle in the presence of the electric field, so that $H|\,\varepsilon\rangle = \varepsilon|\varepsilon\rangle$.

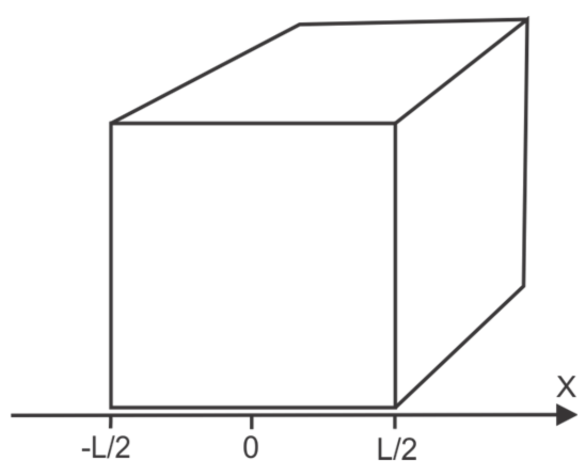
a. Write down the eigenvalue equation obeyed by the energy eigenfunctions $\varphi_\varepsilon(x)$ in the real-space or position representation.

b. Write down the eigenvalue equation obeyed by the energy eigenfunctions $\varphi_\varepsilon(k)$ or $\varphi_\varepsilon(p)$ in the wavevector or momentum-space representation ($p = \hbar k$).

c. Solve the energy eigenvalue equation of the last part to obtain, up to a normalization constant, the energy eigenfunctions $\varphi_\varepsilon(k)$ or $\varphi_\varepsilon(p)$ **in the momentum-space or wavevector representation**. For what energies ε are your solutions acceptable? What is the degeneracy of each energy eigenvalue ε?

d. By considering the Fourier transform, show that the real-space eigenfunctions $\varphi_\varepsilon(x)$ at different energies are related to one another through a simple spatial shift, i.e., there exists for each ε a constant ℓ_ε having units of length such that $\varphi_\varepsilon(x) = \varphi_{\varepsilon=0}(x - \ell_\varepsilon)$.

A5. Statistical Mechanics: Gas of charged particles



A cubic box of linear size L (with one edge oriented along the x-axis) contains N classical particles of charge +e as well as N classical particles of charge –e. The box is in an electric field E directed in the positive x direction. The particles are in thermal equilibrium at temperature T. (Neglect the Coulomb interaction between the particles!)

a) Determine the (thermal average of the) total electric dipole moment of the system as a function of temperature and electric field.
b) Discuss the limits of high and low temperatures ($k_BT >> ELe$ or $k_BT << ELe$)

A6. In the following problem you may ignore the Coulomb interaction and work in the z-basis.

Suppose a positron and electron interact and form a spin singlet state such that the wave function is

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) \text{ at time } t = 0.$$

If their separation is large we can ignore the spin-spin interaction. If the pair is in a static magnetic field $\vec{B} = B_0\hat{z}$, then the Hamiltonian is given by $\hat{H} = \omega_0(\hat{S}_{1z} - \hat{S}_{2z})$, where ω_0 is related to B_0.

(a) If the elements in the z-basis are ordered as $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$, find the matrix representing the Hamiltonian in the z-basis?

(b) Determine the matrix representing the time-evolution operator in the z-basis and find the wave function $|\psi(t)\rangle$ as a function of time.

(c) Does the magnetic field cause mixing of the singlet and triplet states? If so which ones, and show them explicitly.

(d) Calculate the probability that at time t a measurement of the spin states S_{1y} and S_{2y} will yield $+\hbar/2$ for both particles.

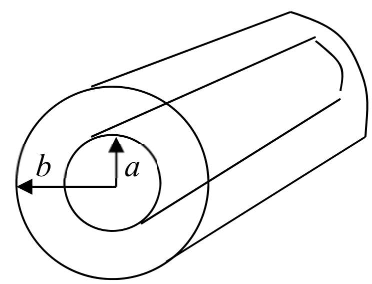
Ph.D. QUALIFYING EXAMINATION – PART B

Wednesday, January 16, 2019, 1:00 – 5:00 P.M.

Work each problem on a separate sheet(s) of paper and put your identifying number on each page. Do not use your name. Each problem has equal weight. A table of integrals can be used. Some physical constants and mathematical definitions will be provided if needed.

B1. Consider a thin disk of radius R composed of two homogeneous halves connected along a diameter of the disk. If one half has density ρ and the other has density 2ρ, find the expression for the Lagrangian when the disk rolls without slipping along a horizontal surface in terms of system parameters. What are the equilibrium points of the system and the frequency of small oscillations around the stable equilibrium(s)?

B2. A long coaxial cable carries a uniform volume charge density ρ on the inner cylinder (radius a), and a uniform surface charge density on the outer cylindrical shell (radius b). This surface charge is negative and of just the right magnitude so that the cable as a whole is electrically neutral. Assume cylindrical coordinates (s, ϕ, z) since ρ is the volume charge density.



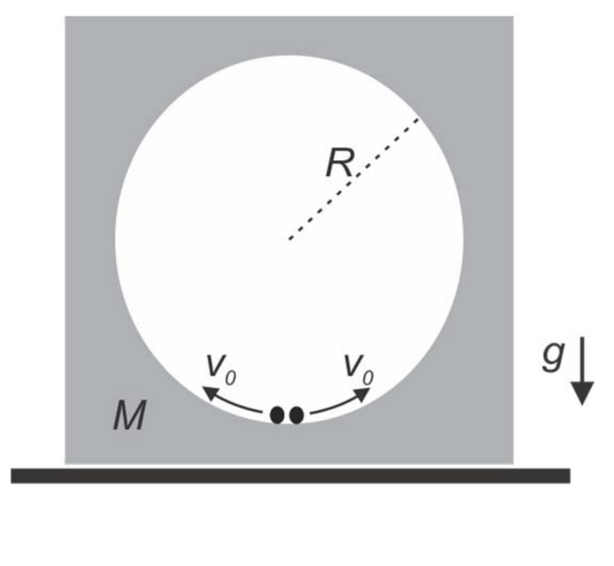
a) Use Gauss's Law to determine the electric field in each of the three regions: inside the inner cylinder $(s < a)$, between the cylinders $(a < s < b)$, outside the cable $(s > b)$.

b) Determine the electrostatic potential in each of the three regions listed above.

c) Determine the energy per unit length stored in the cable.

B3. **Classical Mechanics: Beads in circular hole**

A large block of mass M is at rest on a horizontal surface. The block contains a circular hole of radius R that forms a frictionless track as shown in the picture. Two beads, each of mass $m,$ are launched at the bottom of the track with initial speed v_0 in opposite directions.
Find the maximum value of v_0 as a function of m, M and R such that the large block is not lifted upwards during the motion of the beads.



B4. Consider a pair of identical two-level quantum subsystems, each with energy splitting Δ. For subsystem $i = 1,2$ let $\{|n_i\rangle_i\} = \{|0\rangle_i, |1\rangle_i\}$ denote a basis of unperturbed eigenstates with energies $\varepsilon_{n_i} = n_i\Delta$. Define for each subsystem a "lowering" operator $a_i = |0\rangle_i\langle 1|_i$ that has the obvious action $a_i|1\rangle_i = |0\rangle_i$ and $a_i|0\rangle_i = 0$.

a) Construct the 2×2 matrices representing a_i, a_i^+, and $N_i = a_i^+ a_i$ in the representation of basis states $|n_i\rangle_i$ for the i-th subsystem. Here, a_i^+ is the adjoint of a_i.

b) Suppose this pair of two-level systems interact with each other, with a total Hamiltonian

$$H = a_1^+ a_1 \Delta + a_2^+ a_2 \Delta + \gamma(a_1^+ a_2 + a_2^+ a_1)$$

in which γ is a positive coupling constant. Show that the total number $N = N_1 + N_2$ of "excitations" in the system is a quantum mechanical constant of the motion under the evolution generated by H.

c) Determine the energy eigenvalues and eigenstates for this pair of interacting two-level systems. Express your answers in terms of two-particle basis states of the form $|n_1, n_2\rangle$, in which $n_i \in \{0,1\}$ denotes the number of excitations in the i-th two-level system.

B5. Flux density measuring mean solar electromagnetic radiation (solar irradiance) per unit area at Earth's orbit is $\Phi_0 \approx 1360 W/m^2$. A fraction (called Bond albedo) $\varepsilon \approx 0.306$ of this flux is immediately reflected back to space. The remaining flux is distributed over the entire surface of the planet (via e.g. revolution of the Earth).

(a) Assuming the Earth can be approximated as a black body, find its flux (energy emitted per unit of area, per unit of time) Φ_1 and the equilibrium temperature T_1. Stefan-Boltzmann constant σ is equal to $5.67 \times 10^{-8} W/m^2K^4$.

(b) Now let us account for the greenhouse effects by the atmosphere. Assume a thin layer of greenhouse gases is (i) completely transparent to the incoming solar radiation, (ii) absorbs the entire 100% of the (blackbody) radiation coming from the Earth, and (iii) re-emits 50% of the absorbed energy back down to the Earth and the remaining 50% into space. Find the new value Φ_2 of flux that reaches the surface in terms of Φ_1.

(c) Based on Φ_2, find the new equilibrium temperature T_2, which now includes the greenhouse effect discussed in (b). Compare your result in (a).

B6. Nearly 200 years ago (1839), Gauss presented a proof that the sources of the magnetic field of the Earth had to be interior to its surface. The proof is an application of potential theory, therefore this problem can be thought of as an electrostatics problem.

a) Imagine a thin spherical shell whose inner radius is the radius of the Earth and whose outer radius encompasses a layer of the atmosphere but not the ionosphere, therefore there are no current sources in the shell [the ionosphere introduces a very small perturbation anyway]. Show that in this layer we can write the magnetic field as $\vec{B} = \vec{\nabla}\Phi$, and that the potential Φ satisfies the Laplace equation.

Since the general solution for the potential is $\Phi(r,\theta,\phi) = \sum_{l,m}(A_{lm}r^l + B_{lm}r^{-(l+1)})Y_{lm}(\theta,\phi)$, the coefficients A_{lm} and B_{lm} can be determined from measurements of the horizontal and vertical components of the magnetic field $\vec{B}$ on the surface of the Earth. The result of Gauss was that there were no A_{lm}'s (84 points spaced at 30° of longitude along 7 small circles of fixed latitude).

b) Write down the most general solution for the potential Φ outside a sphere of radius equal to the radius of the Earth, assuming the sources are inside the sphere. Show that the result is of the form found by Gauss.